

Covariant approach to the no-ghost theorem in massive gravity

Taichiro Kugo^{1,*} and Nobuyoshi Ohta^{2,*}

¹*Department of Physics and Maskawa Institute for Science and Culture, Kyoto Sangyo University, Kyoto 603-8555, Japan*

²*Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*

*E-mail: kugo@cc.kyoto-su.ac.jp, ohtan@phys.kindai.ac.jp

Received January 20, 2014; Accepted March 10, 2014; Published April 14, 2014

.....
We discuss the no-ghost theorem in massive gravity in a covariant manner. Using the Becchi-Rouet-Stora-Tyutin (BRST) formalism and Stückelberg fields, we first clarify how the Boulware-Deser ghost decouples in massive gravity theory with the Fierz-Pauli mass term. Here we find that the crucial point in the proof is that there is no higher (time) derivative for the Stückelberg “scalar” field. We then analyze the nonlinear massive gravity proposed by de Rham, Gabadadze, and Tolley, and show that there is no ghost for general admissible backgrounds. In this process, we find a very nontrivial decoupling limit for general backgrounds. We end the paper by demonstrating the general results explicitly in a nontrivial example where there apparently appear higher time derivatives for the Stückelberg scalar field, but show that this does not introduce the ghost into the theory.
.....

Subject Index B33, B39, E00, E03

1. Introduction

Recently, there has been renewed interest in the search for the modification of gravity at large distances by adding mass terms for the graviton. The motivation for this comes from both theoretical and observational considerations.

On the theoretical side, it is interesting to explore the possibility of formulating a massive spin-2 field theory. In general relativity, which describes the massless spin-2 field, the four constraints of the theory together with the invariance under the four general coordinate transformations remove eight of the modes from the ten degrees of freedom in the metric, and the number of propagating modes reduces to the physical two modes of the massless graviton. When the mass term is added, the four constraints remove four propagating modes, but the general covariance is broken. Thus there remain six degrees of freedom in general. Five of these are the modes of the massive spin-2 graviton, but it turns out that the sixth scalar mode is a ghost with a negative metric. A unique mass term that does not contain this ghost is known as a Fierz-Pauli mass term [1].

However, it has been pointed out that this theory suffers from the problem that the helicity-0 state couples to the trace of the matter stress-energy tensor with the same strength as the helicity-2 state [2, 3]. This means that this massive gravity does not continuously reduce to general relativity in the massless limit. This is called the van Dam-Veltman-Zakharov discontinuity. It was then argued by Vainshtein that the discontinuity could be avoided by the nonlinear interaction [4]. Unfortunately, the very nonlinearity that cures the discontinuity problem re-introduces the ghost into the theory, known

as the Boulware–Deser (BD) ghost [5]. Consistent extension of the mass term to the nonlinear level is a major theoretical problem.

On the observational side, the recent discovery of the accelerating expansion of the universe suggests the modification of either the gravity side or the matter side of the Einstein equation. A simple extension would be to introduce the cosmological constant, which must be extremely tiny to account for the current observation. Another modification on the gravity side is to consider massive gravity, because cosmological solutions with an accelerated expansion are expected if the gravity becomes weaker on the larger scale.

Recently, an interesting proposal to extend the work of Fierz–Pauli [1] to the nonlinear level has been made by de Rham, Gabadadze, and Tolley (dRGT) [6,7] by generalizing the effective field theory approach [8]. It was first shown that there is no BD ghost to all orders in the decoupling limit (defined in the flat space). It was then argued that this formulation of massive gravity has no ghost at the nonlinear level [9–20]. Using the noncovariant Arnowitt–Deser–Misner (ADM) decomposition, it is shown that the mass term introduces nonlinear terms for the shifts so that these do not produce any constraint, but the lapse function remains linear and we are left with one constraint instead of the four in general relativity. Thus, in this noncovariant approach, we have six degrees of freedom for the propagating modes from the spatial metric g_{ij} , but one of them is removed by the above constraint from the lapse, leaving the correct five degrees of freedom for a massive spin-2 without ghost. The proof is valid to the full nonlinear level, but it is based on a noncovariant formulation and is very indirect one just counting the degrees of freedom. So it is unclear why there remains such a linear lapse variable in the dRGT massive gravity.

One interesting approach is to introduce Stückelberg fields, which recover the general coordinate invariance [8,10,15] and additional gauge invariance. Here, again using ADM decomposition, it is shown that we get the correct five physical degrees of freedom in the theory. This is again a noncovariant approach.

Another covariant approach to the problem is presented in Ref. [17], which again uses constraints to remove the degrees of freedom, but the proof is not completed for general mass terms, in particular in the presence of the cubic mass term.

In this paper, we use the covariant approach based on the Becchi–Rouet–Stora–Tyutin (BRST) formalism and Stückelberg fields to show explicitly the cancellation of the ghost degrees of freedom for arbitrary backgrounds, and clarify the structure of the theory. We show that we have 10 degrees of freedom from the graviton, 4 from the Stückelberg vector, and 1 from the Stückelberg scalar field, minus 4×2 from the vector Faddeev–Popov ghost and anti-ghost, minus 1×2 from the scalar Faddeev–Popov ghost and anti-ghost. This leaves us with 5 degrees of freedom, the right number for massive spin-2 fields. An important point is that there is no higher derivative term for the kinetic term of the Stückelberg scalar field, which (if present) introduces an additional ghost degree of freedom unless the mass term is judiciously chosen. It has been shown for several cases that there is no such higher-order term, or it is present but in a harmless way in the ADM formulation [10]. However, it is not clear if this is true in general and for arbitrary backgrounds. Here we give the complete proof of the absence of the ghost with all possible mass terms and on general backgrounds in a covariant manner.

This paper is organized as follows. In Sect. 2, in order to clarify how our approach works, we show in detail how the BD ghost is decoupled in the simple theory with the Fierz–Pauli mass term in our formulation. Since it is easy to do this for arbitrary dimensions, we discuss the problem in general dimensions D . First, in Sect. 2.1, we discuss the streamlined proof of the no-ghost theorem using

the Stückelberg fields and BRST formalism in this theory. Here we count the degrees of freedom, and clarify that the necessary and sufficient condition for the theory to be ghost-free is that there is no higher (time) derivative of the Stückelberg fields. The discussion is completed in Sect. 2.2, where we compute the propagators for all the fields in the theory and show that all the ghost degrees of freedom cancel against the Faddeev–Popov ghosts, and there remain only physical $\frac{(D-2)(D+1)}{2}$ (five for four dimensions) degrees of freedom for spin-2.

In Sect. 3, we come to the main theme of this paper: to prove the no-ghost theorem in nonlinear massive gravity on arbitrary backgrounds. In Sect. 3.1, we first discuss how to diagonalize the general background metric in order to properly take its square root, which is necessary to rewrite the mass term suitable for examining the spectrum. We then use this result in Sect. 3.2 to compute the generating function of the mass terms. In Sect. 3.3, we find that there is an important hidden $U(1)$ gauge invariance that ensures the decoupling of the ghost. In this process, we find that the way how to introduce the Stückelberg fields in a general background is significantly modified from its counterpart for a flat background, and the associated decoupling limit is also quite nontrivial. We then show that there are no higher derivative terms for the Stückelberg fields. Combined with the above result in the Fierz–Pauli mass term, this implies that there remains no BD ghost in this massive gravity. In Sect. 4, we go on to discuss an explicit and nontrivial example for a background metric with shift. We show that, naively, higher time derivatives on the Stückelberg scalar field seem to appear, but our definition of the Stückelberg fields avoids the trouble, so that there is no BD ghost in the theory.

2. Absence of ghosts in the Fierz–Pauli mass term

In this section, we first discuss the no-ghost theorem in massive gravity with the Fierz–Pauli mass term in arbitrary dimensions D . Let us consider the action

$$S = \frac{1}{\kappa^2} \int d^D x \sqrt{-g} \left[R - \frac{m^2}{4} (h_{\mu\nu}^2 - a h^2) \right], \quad (2.1)$$

where κ^2 is the D -dimensional gravitational constant, m and a are constants. Here $h_{\mu\nu}$ is the fluctuation of the metric around the background spacetime

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \quad (2.2)$$

and $h \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$. We use the conventions in Ref. [21] and set $\kappa = 1$ henceforth. In the rest of this section, we consider the flat background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ for simplicity.

At first sight, one expects that this theory contains $\frac{(D-2)(D+1)}{2}$ (five for four dimensions) degrees of freedom, corresponding to the massive spin-2 field. However, it has been shown that this massive gravity contains an additional (sixth in four dimensions) degree of freedom unless $a = 1$, known as the BD ghost [5]. We first recapitulate how to understand this situation.

2.1. Stückelberg fields and BRST formalism

Because of the presence of the mass term in (2.1), there is no invariance under the general coordinate transformation. We can recover the invariance by introducing the Stückelberg fields, as was shown by Arkani-Hamed, Georgi, and Schwartz [8]. In their formulation, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is replaced by $h_{\mu\nu} = g_{\mu\nu} - f_{\mu\nu}$, where $f_{\mu\nu}$ is the fiducial metric given as the general coordinate transformation of the flat metric $\eta_{\mu\nu}$ using Stückelberg fields (see the precise definition given later in Sect. 3). This

replacement reduces at the linearized level simply to [22]

$$h_{\mu\nu} \Rightarrow h_{\mu\nu} - \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{2}{m^2}\partial_\mu\partial_\nu\pi, \quad (2.3)$$

where m is a mass scale. The metric is invariant under the transformation

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \delta A_\mu = m\xi_\mu + \partial_\mu \Lambda, \quad \delta\pi = m\Lambda. \quad (2.4)$$

In order to quantize the theory, we gauge fix the theory and introduce the Faddeev–Popov ghosts and anti-ghosts corresponding to the invariance (2.4). They are the vector ghost c_μ and anti-ghost \bar{c}_μ for ξ_μ , and the scalar ghost c and anti-ghost \bar{c} for Λ .

Now the physical degrees of freedom in the theory are counted as $\frac{D(D+1)}{2}$ (10 for four dimensions) from $h_{\mu\nu}$, D (4 for four dimensions) from A_μ and 1 from π , minus $D \times 2$ (4×2 for four dimensions) from the vector ghost and anti-ghost, minus 2 from the scalar ghost and anti-ghost. This leaves us with $\frac{(D-2)(D+1)}{2}$ (5 for four dimensions) degrees of freedom, the right number for massive spin-2 fields. However, this cannot be true in general. It is known that the theory (2.1) describes $\frac{D(D-1)}{2}$ (6 for four dimensions) degrees of freedom unless $a = 1$ and one of them is a ghost. What is wrong with this counting then?

We can see the origin of the problem if we substitute (2.3) into the action: The quadratic part of the mass terms of the Lagrangian in (2.1) takes the form

$$\begin{aligned} & -\frac{m^2}{4}(h_{\mu\nu}^2 - ah^2) - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - (1-a)(\partial_\mu A^\mu)^2 - (mA_\mu - \partial_\mu\pi)(\partial_\nu h^{\mu\nu} - \partial^\mu h) \\ & - m(a-1)h\partial_\mu A^\mu + (a-1)h\Box\pi + \frac{2(1-a)}{m}\partial_\mu A^\mu\Box\pi - \frac{(1-a)}{m^2}(\Box\pi)^2. \end{aligned} \quad (2.5)$$

The last term here indicates that the field π has two degrees of freedom unless $a = 1$, so the above counting is not correct. If and only if $a = 1$, the above counting is correct and we are left with $\frac{(D-2)(D+1)}{2}$ (5 for four dimensions) degrees of freedom. Note that the terms involving only π vanish in this case, and this corresponds to the requirement of no ghost in the decoupling limit. The mixing with the metric fluctuation gives the dynamics to π . This can be checked by taking the determinant of the kinetic matrix (containing only second derivatives) to see if it does not vanish identically.

Alternatively, we can see that the shift $h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{2}{D-2}\eta_{\mu\nu}\pi$ in the Einstein term

$$\mathcal{L}_{E,2} = \frac{1}{4}h^{\mu\nu}\left[\partial_\mu\partial_\nu h - \partial_\mu h_\nu - \partial_\nu h_\mu + \Box h_{\mu\nu} + \eta_{\mu\nu}(\partial_\lambda h^\lambda - \Box h)\right] \quad (2.6)$$

cancels the mixing and produces a normal kinetic term for π . Here we have defined

$$h_\mu = \partial^\nu h_{\mu\nu}, \quad h = h^\mu_\mu. \quad (2.7)$$

At this stage, we have

$$\begin{aligned} \mathcal{L}_{E,2} + \mathcal{L}_{\text{mass}} & \rightarrow \mathcal{L}_{E,2} - \frac{m^2}{4}(h_{\mu\nu}^2 - h^2) + \frac{D-1}{D-2}m^2 h\pi + \frac{D(D-1)}{(D-2)^2}m^2\pi^2 \\ & - \frac{1}{4}F_{\mu\nu}(A)^2 - mA^\mu\left(h_\mu - \partial_\mu h - 2\frac{D-1}{D-2}\partial_\mu\pi\right) + \frac{D-1}{D-2}\pi\Box\pi. \end{aligned} \quad (2.8)$$

Here and henceforth in this section, all the $h_{\mu\nu}$ and h denote the new gravity fields after the above shifting:

$$h_{\mu\nu} = h_{\mu\nu}^{\text{original}} - \frac{2}{D-2}\eta_{\mu\nu}\pi, \quad h \equiv \eta^{\mu\nu}h_{\mu\nu} = h^{\text{original}} - \frac{2D}{D-2}\pi. \quad (2.9)$$

We now discuss the gauge fixing of the theory and examine explicitly what spectrum we have. In order to resolve the field mixing terms, we adopt the so-called R_ξ gauges. The gauge fixing and

Faddeev–Popov terms are concisely written as

$$\begin{aligned}\mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} = & -i\delta_{\text{B}} \left[\bar{c}^{\mu} \left(h_{\mu} - x\partial_{\mu}h - \alpha mA_{\mu} + \frac{\alpha}{2}B_{\mu} \right) \right] \\ & - i\delta_{\text{B}} \left[\bar{c} \left\{ \partial A - m\beta(yh + z\pi) + \frac{\beta}{2}B \right\} \right],\end{aligned}\quad (2.10)$$

where α, β, x, y, z are gauge parameters, and the (fermionic) BRST transformations are defined as

$$\begin{aligned}\delta_{\text{B}}h_{\mu\nu} &= \partial_{\mu}c_{\nu} + \partial_{\nu}c_{\mu} - \frac{2}{D-2}\eta_{\mu\nu}mc, \quad \delta_{\text{B}}A_{\mu} = mc_{\mu} + \partial_{\mu}c, \quad \delta_{\text{B}}\pi = mc, \\ \delta_{\text{B}}c_{\mu} &= c^{\rho}\partial_{\rho}c_{\mu}, \quad \delta_{\text{B}}\bar{c}_{\mu} = iB_{\mu}, \quad \delta_{\text{B}}B_{\mu} = 0, \quad \delta_{\text{B}}c = c^{\rho}\partial_{\rho}c, \quad \delta_{\text{B}}\bar{c} = iB, \quad \delta_{\text{B}}B = 0.\end{aligned}\quad (2.11)$$

Using (2.10), we find the gauge fixing and Faddeev–Popov terms as follows:

$$\begin{aligned}\mathcal{L}_{\text{GF}} &= B^{\mu} (h_{\mu} - x\partial_{\mu}h - \alpha mA_{\mu}) + \frac{\alpha}{2}B_{\mu}^2 + B (\partial A - m\beta(yh + z\pi)) + \frac{\beta}{2}B^2 \\ &= -\frac{1}{2\alpha}[h_{\mu} - x\partial_{\mu}h - \alpha mA_{\mu}]^2 - \frac{1}{2\beta}[\partial_{\mu}A^{\mu} - m\beta(yh + z\pi)]^2 + \frac{\alpha}{2}B_{\mu}^2 + \frac{\beta}{2}B^2,\end{aligned}\quad (2.12)$$

$$\begin{aligned}\mathcal{L}_{\text{FP}} &= i\bar{c}^{\mu} \left[\partial_{\mu}\partial^{\nu}c_{\nu} + \square c_{\mu} - m\frac{2}{D-2}\partial_{\mu}c - x\partial_{\mu} \left(2\partial^{\nu}c_{\nu} - \frac{2D}{D-2}mc \right) - \alpha m^2c_{\mu} - \alpha m\partial_{\mu}c \right] \\ &\quad + i\bar{c} \left[m\partial^{\nu}c_{\nu} + \square c - m\beta \left(y(2\partial^{\nu}c_{\nu} - \frac{2D}{D-2}mc) + zmc \right) \right].\end{aligned}\quad (2.13)$$

Here B'_{μ} and B' are the shifted B_{μ} and B fields to complete the squares:

$$\begin{aligned}B'_{\mu} &= B_{\mu} + \alpha^{-1} (h_{\mu} - x\partial_{\mu}h - \alpha mA_{\mu}), \\ B' &= B + \beta^{-1} (\partial A - m\beta(yh + z\pi)).\end{aligned}\quad (2.14)$$

Now we determine the gauge parameters x, y , and z so as to cancel the various field transition terms as follows. Note that this gauge fixing term (2.12) is arranged to cancel the term $-mA_{\mu}h^{\mu}$ in (2.8). In order to cancel the $A_{\mu}\partial^{\mu}h$ term in (2.8) by the corresponding terms from the gauge fixing term (2.12), we should have

$$x + y = 1. \quad (2.15)$$

To cancel the $A_{\mu}\partial^{\mu}\pi$ term in (2.8) by a term from (2.12), we set

$$z = 2\frac{D-1}{D-2}. \quad (2.16)$$

Finally, the $h\pi$ mixing term in (2.8) can be canceled by that from (2.12) by choosing

$$y = \frac{1}{2\beta}. \quad (2.17)$$

The resulting total quadratic Lagrangian is

$$\begin{aligned}\mathcal{L}_t &= \mathcal{L}_{E,2} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{GF}} \\ &= \left(\frac{1}{2} - \frac{x}{\alpha} \right) h\partial_{\mu}h^{\mu} + \left(\frac{1}{2} - \frac{1}{2\alpha} \right) h_{\mu}^2 + \frac{1}{4}h_{\mu\nu}(\square - m^2)h^{\mu\nu} - \left(\frac{1}{4} - \frac{x^2}{2\alpha} \right) h\square h \\ &\quad + \frac{xm^2}{4}h^2 - \frac{1}{4}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})^2 - \frac{1}{2\beta}(\partial A)^2 - \frac{1}{2}\alpha m^2A_{\mu}^2 + \frac{z}{2}\pi[\square - 2\beta wm^2]\pi,\end{aligned}\quad (2.18)$$

with the parameter w denoting

$$w = \frac{x D - 1}{D - 2}. \quad (2.19)$$

The action takes a simple form for $\alpha = \beta = 1$, in which case $x = 1/2$ and all the fields have the same mass m^2 .

2.2. Propagators

2.2.1. Tensor propagator. To calculate the propagators, let us introduce projection operators in momentum space:

$$\begin{aligned} d_{\mu\nu} &\equiv \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, & e_{\mu\nu} &\equiv \frac{p_\mu p_\nu}{p^2}, \\ I_{\mu\nu,\rho\sigma} &\equiv \frac{1}{2} \left(d_{\mu\rho} d_{\nu\sigma} + d_{\mu\sigma} d_{\nu\rho} - \frac{2}{D-1} d_{\mu\nu} d_{\rho\sigma} \right), \\ II_{\mu\nu,\rho\sigma} &\equiv \frac{1}{2} [d_{\mu\rho} e_{\nu\sigma} + d_{\mu\sigma} e_{\nu\rho} + (\mu \leftrightarrow \nu)], \end{aligned} \quad (2.20)$$

which satisfy

$$\begin{aligned} d_{\mu,\rho} d^\rho_\nu &= d_{\mu\nu}, & d_{\mu,\rho} e^\rho_\nu &= e_{\mu,\rho} d^\rho_\nu = 0, & e_{\mu,\rho} e^\rho_\nu &= e_{\mu\nu}, \\ I_{\mu\nu,\alpha\beta} I^{\alpha\beta}_{\rho\sigma} &= I_{\mu\nu,\rho\sigma}, & II_{\mu\nu,\alpha\beta} II^{\alpha\beta}_{\rho\sigma} &= II_{\mu\nu,\rho\sigma}, \\ I_{\mu\nu,\alpha\beta} II^{\alpha\beta}_{\rho\sigma} &= II_{\mu\nu,\alpha\beta} I^{\alpha\beta}_{\rho\sigma} = 0, \\ I_{\mu\nu,\rho\sigma} d^{\rho\sigma} &= I_{\mu\nu,\rho\sigma} e^{\rho\sigma} = 0, & II_{\mu\nu,\rho\sigma} d^{\rho\sigma} &= II_{\mu\nu,\rho\sigma} e^{\rho\sigma} = 0. \end{aligned} \quad (2.21)$$

Note that

$$\begin{aligned} h \partial_\mu h^\mu &= -\frac{1}{2} h^{\mu\nu} p^2 (d_{\mu\nu} e_{\rho\sigma} + e_{\mu\nu} d_{\rho\sigma} + 2e_{\mu\nu} e_{\rho\sigma}) h^{\rho\sigma}, \\ h_\mu h^\mu &= +\frac{1}{2} h^{\mu\nu} p^2 (II_{\mu\nu,\rho\sigma} + 2e_{\mu\nu} e_{\rho\sigma}) h^{\rho\sigma}, \\ h_{\mu\nu} \square h^{\mu\nu} &= -h^{\mu\nu} p^2 \left(I_{\mu\nu,\rho\sigma} + II_{\mu\nu,\rho\sigma} + \frac{1}{D-1} d_{\mu\nu} d_{\rho\sigma} + e_{\mu\nu} e_{\rho\sigma} \right) h^{\rho\sigma}, \\ h \square h &= -h^{\mu\nu} p^2 (d_{\mu\nu} d_{\rho\sigma} + (e_{\mu\nu} d_{\rho\sigma} + d_{\mu\nu} e_{\rho\sigma}) + e_{\mu\nu} e_{\rho\sigma}) h^{\rho\sigma}. \end{aligned} \quad (2.22)$$

Using these, we find that the quadratic term in the gravity field $h_{\mu\nu}$ is written in the form

$$\frac{1}{2} h^{\mu\nu} \mathcal{Q}_{\mu\nu,\rho\sigma} h^{\rho\sigma}, \quad (2.23)$$

where

$$\mathcal{Q}_{\mu\nu,\rho\sigma} = \mathcal{A} I_{\mu\nu,\rho\sigma} + \mathcal{B} II_{\mu\nu,\rho\sigma} + \mathcal{C} d_{\mu\nu} d_{\rho\sigma} + \mathcal{D} (e_{\mu\nu} d_{\rho\sigma} + d_{\mu\nu} e_{\rho\sigma}) + \mathcal{E} e_{\mu\nu} e_{\rho\sigma}, \quad (2.24)$$

with

$$2\mathcal{A} = -(p^2 + m^2), \quad 2\mathcal{B} = -\frac{p^2 + \alpha m^2}{\alpha}, \quad (2.25)$$

$$\mathcal{C} = \frac{D-2}{2(D-1)} (p^2 + 2\beta w m^2) - \frac{x^2}{\alpha} (p^2 + \alpha\beta m^2),$$

$$\mathcal{D} = \frac{x}{2\alpha\beta} (p^2 + \alpha\beta m^2), \quad \mathcal{E} = -\frac{1}{4\alpha\beta^2} (p^2 + \alpha\beta m^2),$$

$$\mathcal{D}^2 - \mathcal{E}\mathcal{C} = \frac{D-2}{8\alpha\beta^2(D-1)} (p^2 + \alpha\beta m^2)(p^2 + 2\beta w m^2). \quad (2.26)$$

The propagator \mathcal{P} ,

$$\mathcal{P}_{\mu\nu,\rho\sigma} = \alpha I_{\mu\nu,\rho\sigma} + \beta II_{\mu\nu,\rho\sigma} + \gamma d_{\mu\nu}d_{\rho\sigma} + \delta (e_{\mu\nu}d_{\rho\sigma} + d_{\mu\nu}e_{\rho\sigma}) + \varepsilon e_{\mu\nu}e_{\rho\sigma}, \quad (2.27)$$

is given by the inverse of the kinetic operator:

$$\mathcal{P}_{\mu\nu,\alpha\beta} \mathcal{Q}^{\alpha\beta}_{\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) = I_{\mu\nu,\rho\sigma} + II_{\mu\nu,\rho\sigma} + \frac{1}{D-1} d_{\mu\nu}d_{\rho\sigma} + e_{\mu\nu}e_{\rho\sigma}. \quad (2.28)$$

This condition requires

$$\begin{aligned} A\alpha = 1, \quad B\beta = 1, \quad (D-1)C\gamma + D\delta &= \frac{1}{D-1}, \\ (D-1)C\delta + D\varepsilon = 0, \quad (D-1)D\gamma + E\delta &= 0, \quad (D-1)D\delta + E\varepsilon = 1, \end{aligned} \quad (2.29)$$

so that we find

$$\alpha = \frac{1}{A} = -\frac{2}{p^2 + m^2}, \quad \beta = \frac{1}{B} = -\frac{2\alpha}{p^2 + \alpha m^2}, \quad (2.30)$$

$$\gamma = -\frac{E}{(D-1)(D^2 - EC)} = \frac{2}{(D-1)(D-2)} \cdot \frac{1}{p^2 + 2\beta w m^2}, \quad (2.31)$$

$$\delta = \frac{D}{(D-1)(D^2 - EC)} = \frac{4\beta x}{D-2} \cdot \frac{1}{p^2 + 2\beta w m^2}, \quad (2.32)$$

$$\varepsilon = -\frac{C}{D^2 - EC} = \frac{8\beta^2 x^2 (D-1)}{D-2} \cdot \frac{1}{p^2 + 2\beta w m^2} - \frac{4\alpha\beta^2}{p^2 + \alpha\beta m^2}. \quad (2.33)$$

We see that most of the terms have gauge-dependent masses, which should cancel with the Faddeev–Popov ghost.

2.2.2. Faddeev–Popov ghost propagator. The kinetic term of the Faddeev–Popov ghosts is:

$$i \begin{pmatrix} \bar{c}^\mu & \bar{c} \end{pmatrix} \begin{pmatrix} -(p^2 + \alpha m^2)d_{\mu\nu} - 2y(p^2 + \alpha\beta m^2)e_{\mu\nu} & imp_\mu(\alpha - 2w) \\ 0 & -p^2 - 2\beta w m^2 \end{pmatrix} \begin{pmatrix} c^\nu \\ c \end{pmatrix}. \quad (2.34)$$

We find the propagator from the inverse of this:

$$\begin{pmatrix} \bar{c}_\mu & \bar{c} \end{pmatrix} \begin{pmatrix} -\frac{d_{\mu\nu}}{p^2 + \alpha m^2} - \beta \frac{e_{\mu\nu}}{p^2 + \alpha\beta m^2} & \frac{-ip_\mu}{p^2 + \alpha m^2}(\alpha - 2w) \frac{\beta m}{p^2 + 2\beta w m^2} \\ 0 & -\frac{1}{p^2 + 2\beta w m^2} \end{pmatrix} \begin{pmatrix} c_\nu \\ c \end{pmatrix}. \quad (2.35)$$

2.2.3. Vector propagator. The kinetic term of the vector field A_μ is given by

$$\begin{aligned} & -\frac{1}{2} [(p^2 + \alpha m^2)\eta_{\mu\nu} - (1 - \beta^{-1})p_\mu p_\nu] \\ & = -\frac{1}{2} [(p^2 + \alpha m^2)d_{\mu\nu} + \beta^{-1}(p^2 + \alpha\beta m^2)e_{\mu\nu}], \end{aligned} \quad (2.36)$$

whose inverse gives the vector propagator:

$$\begin{aligned} \langle A_\mu A_\nu \rangle &= -\frac{d_{\mu\nu}}{p^2 + \alpha m^2} - \frac{\beta e_{\mu\nu}}{p^2 + \alpha\beta m^2} \\ &= \frac{\eta_{\mu\nu} + \frac{p_\mu p_\nu}{\alpha m^2}}{-p^2 - \alpha m^2} - \left(\frac{p_\mu p_\nu}{\alpha m^2} \right) \frac{1}{-p^2 - \alpha\beta m^2}. \end{aligned} \quad (2.37)$$

Note that the massless singularities contained in $d_{\mu\nu}$ and $e_{\mu\nu}$ have actually been canceled in this vector propagator. This should be so since those singularities are of course not physical but an artifact

of our computational device using projection operators. The same cancellations of massless singularities have also occurred in the above tensor propagators, which the reader can confirm by using the above expressions for the tensor propagator.

Summarizing, we have the following propagators after suitable normalization:

$h_{\mu\nu}$ -sector:

$$\begin{aligned}
 h_{\text{TT}} : & \text{ transverse-traceless } \frac{(D-2)(D+1)}{2}\text{-modes} & -\frac{1}{p^2 + m^2}, \\
 h_{\text{LT}} : & \text{ longitudinal-transverse } (D-1)\text{-modes} & -\frac{1}{p^2 + \alpha m^2}, \\
 h_{\text{LL}} + h : & \text{ LL and trace } (1+1)\text{-modes} & -\frac{1}{p^2 + \alpha\beta m^2}, \\
 & & -\frac{1}{p^2 + 2\beta w m^2};
 \end{aligned} \tag{2.38}$$

A_μ - π -sector:

$$\begin{aligned}
 A_{\text{T}} : & \text{ massive vector } (D-1)\text{-modes} & -\frac{1}{p^2 + \alpha m^2}, \\
 A_{\text{L}} : & \text{ longitudinal 1-mode} & -\frac{1}{p^2 + \alpha\beta m^2}, \\
 \pi : & \text{ scalar 1-mode} & -\frac{1}{p^2 + 2\beta w m^2};
 \end{aligned} \tag{2.39}$$

Faddeev–Popov ghost sector:

$$\begin{aligned}
 \bar{c}_{\text{T}}, c_{\text{T}} : & \text{ massive } 2(D-1)\text{-modes} & -\frac{1}{p^2 + \alpha m^2}, \\
 \bar{c}_{\text{L}}, c_{\text{L}} : & \text{ longitudinal } (1+1)\text{-modes} & -\frac{1}{p^2 + \alpha\beta m^2}, \\
 \bar{c}, c : & \text{ scalar } (1+1)\text{-modes} & -\frac{1}{p^2 + 2\beta w m^2}.
 \end{aligned} \tag{2.40}$$

We see that almost all modes cancel out with the Faddeev–Popov ghosts, and we are left with $\frac{(D-2)(D+1)}{2}$ (five for four dimensions) modes of the symmetric transverse-traceless tensor $h_{\mu\nu}$ with mass m .

3. Absence of ghosts in nonlinear massive gravity

We now consider four-dimensional theory for nonlinear massive gravity as formulated by dRGT [6,7]. For simplicity, here we discuss only four-dimensional theory, but the generalization to arbitrary dimensions is straightforward. The action is given by [6,7]

$$S = \int d^4x \sqrt{-g} [R + m^2 \mathcal{L}_{\text{mass}}], \tag{3.1}$$

where $\mathcal{L}_{\text{mass}}$ is given by

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}[(K_\mu^\mu)^2 - K_\mu^\nu K_\nu^\mu] + \frac{c_3}{3!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\sigma} K_\alpha^\mu K_\beta^\nu K_\gamma^\rho + \frac{c_4}{4!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} K_\alpha^\mu K_\beta^\nu K_\gamma^\rho K_\delta^\sigma. \tag{3.2}$$

Here c_3, c_4 are parameters and

$$K^\mu{}_\nu = \delta^\mu{}_\nu - \gamma^\mu{}_\nu, \quad \gamma^\mu{}_\nu = \sqrt{g^{\mu\sigma} f_{\sigma\nu}}, \tag{3.3}$$

where $f_{\mu\nu}$ is a fiducial metric that can be chosen to be a flat metric $\eta_{\mu\nu}$. Actually, we would like to keep the general coordinate invariance by introducing the Stückelberg field Y^M . Following Ref. [8],

we set the fiducial metric to

$$f_{\mu\nu} = \partial_\mu Y^M G_{MN} \partial_\nu Y^N. \quad (3.4)$$

Here Y^M is a coordinate in the “target space” and we can set it to

$$Y^M(x) = x^\mu \delta_\mu^M + \phi^M(x), \quad (3.5)$$

obtaining

$$\partial_\mu Y^M = \delta_\mu^M + \partial_\mu \phi^M, \quad (3.6)$$

where μ and M represent the “worldsheet” and “target space” indices, respectively. The original dRGT formulation corresponds to taking the target space metric G_{MN} to be flat Minkowski’s η_{MN} , as we follow henceforth. We then have

$$f_{\mu\nu} = \eta_{\mu\nu} + (\partial_\mu \phi_\nu + \partial_\nu \phi_\mu) + \partial_\mu \phi^\rho \cdot \partial_\nu \phi_\rho, \quad (3.7)$$

where we freely raise and lower the index μ of $\phi^\mu = \phi^M \delta_M^\mu$ by the Minkowski metric: $\phi_\mu = \eta_{\mu\nu} \phi^\nu$, $\phi^\mu = \eta^{\mu\nu} \phi_\nu$. We should note that, although we use the formulation in which the general coordinate invariance is recovered, our following discussions proceed with this choice of fiducial metric; we restrict ourselves to the frame where the Stückelberg fields ϕ_μ have no vacuum expectation value.

We are interested in the question of whether there are higher time-derivative terms in the Stückelberg fields. To study this, we introduce the metric fluctuation around the general background $\bar{g}_{\mu\nu}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (3.8)$$

However, since we are interested only in the question of whether there remains a BD ghost that exists in the Stückelberg modes, we can simply set the graviton fluctuation to zero:

$$h_{\mu\nu} = 0, \quad (3.9)$$

and study the spectrum. What we have to show now is that there are no higher derivative kinetic terms for the Stückelberg fields. If this is confirmed, the preceding discussion shows that we have only five degrees of freedom and there is no BD ghost.

3.1. Diagonalizing the background

The expansion of the square root $\sqrt{\bar{g}^{-1}f}$ around the general background \bar{g} is very complicated in general, if not impossible. For example, one *cannot* simply make an expansion like

$$\sqrt{A+B} \stackrel{?}{=} \sqrt{A} \left(1 + \frac{1}{2} A^{-1} B - \frac{1}{8} (A^{-1} B)^2 + \dots \right), \quad (3.10)$$

unless the matrices A and B commute with each other. We can make a general expansion around a unit matrix as

$$\sqrt{A+B} = \sum_{n=0}^{\infty} {}_n C_{1/2} (A-1+B)^n, \quad (3.11)$$

with a binomial coefficient ${}_n C_{1/2}$, but then the term $(A-1+B)^n$ is not so simple:

$$(A-1+B)^n = (A-1)^n + \sum_{k=1}^n (A-1)^{k-1} B (A-1)^{n-k} + O(B^2), \quad (3.12)$$

because A and B do not commute with each other in general. This expression is too complicated to analyze. Our strategy is then to try to make the background diagonal, in which case we can make a more tractable expansion.

Consider the expression

$$\bar{g}^{\mu\rho} f_{\rho\nu} = \bar{g}^{\mu\rho} \eta_{\rho\sigma} (\delta^\sigma_\nu + \eta^{\sigma\tau} (\partial_\tau \phi_\nu + \partial_\nu \phi_\tau) + \eta^{\sigma\tau} \partial_\tau \phi^\alpha \cdot \partial_\nu \phi_\alpha), \quad (3.13)$$

or

$$\bar{g}^{-1} f = (\bar{g}^{-1} \eta) (1 + \eta^{-1} ((\partial\phi) + (\partial\phi)^T) + \eta^{-1} (\partial\phi) \eta^{-1} (\partial\phi)^T), \quad (3.14)$$

in matrix form.

The c-number part $\bar{g}^{-1} \eta$ can generally be made diagonal by a matrix V . This is true when all the eigenvectors of the 4×4 matrix $\bar{g}^{-1} \eta$ are independent and not degenerate. Degeneracy of the eigenvectors may occur at measure-zero points in the functional space of the background metric $\bar{g}_{\mu\nu}$. Moreover, as we shall see later in an explicit example, we suspect that such degeneracy occurs at the metric $\bar{g}_{\mu\nu}$, which corresponds to a rather singular and unphysical background. Therefore, we confine ourselves to the cases where the matrix $\bar{g}^{-1} \eta$ can be made diagonal.

Let $\alpha(n)$ ($n = 1, 2, 3, 4$) be the roots of the characteristic equation $\det[x1 - \bar{g}^{-1} \eta] = 0$, and V_n be eigenvectors of the matrix $\bar{g}^{-1} \eta$ belonging to the eigenvalue $\alpha(n)$:

$$(\bar{g}^{-1} \eta)^\mu{}_\rho V^\rho_n = \alpha(n) V^\mu_n \quad \text{or} \quad (\bar{g}^{-1} \eta) V_n = \alpha(n) V_n. \quad (3.15)$$

Note that we use *Roman letters* to denote the eigenvector labels in distinction to the original vector indices denoted by Greek letters. Since the matrix $\bar{g}^{-1} \eta$ satisfies

$$(\bar{g}^{-1} \eta) V = V (\alpha(m) \delta_{mn}) \quad \text{for} \quad V \equiv (V_1, V_2, \dots, V_4), \quad (3.16)$$

it is made diagonal as

$$V^{-1} \bar{g}^{-1} \eta V = (\alpha(m) \delta_{mn}) \equiv A^{(0)}. \quad (3.17)$$

Noting that \bar{g} is real symmetric, we can show that the matrix V satisfies

$$V^{-1} = V^T \eta. \quad (3.18)$$

Indeed, using Eq. (3.16) and also its transpose, we can show

$$V^T (\eta \bar{g}^{-1} \eta) V = V^T \eta V (\alpha(m) \delta_{mn}) = (\alpha(m) \delta_{mn}) V^T \eta V. \quad (3.19)$$

If all the eigenvalues are different from one another, this implies that $V^T \eta V$ is diagonal, so that we can realize $V^T \eta V = 1$ by the normalization condition for the eigenvectors. Even if some eigenvalues are degenerate, we can realize it as the orthonormalization condition in each common eigenvalue sector.

Performing a similar transformation to (3.14) with the matrix V , and using the relation (3.18), we find

$$\begin{aligned} V^{-1} (\bar{g}^{-1} f) V &= V^{-1} (\bar{g}^{-1} \eta) V V^{-1} (1 + \eta^{-1} ((\partial\phi) + (\partial\phi)^T) + \eta^{-1} (\partial\phi) \eta^{-1} (\partial\phi)^T) V \\ &= A^{(0)} (1 + V^T ((\partial\phi) + (\partial\phi)^T) V + V^T (\partial\phi) V V^T (\partial\phi)^T V). \end{aligned} \quad (3.20)$$

It is important to notice here that both the “vector” indices μ of ∂_μ and of the Stückelberg field ϕ_μ are commonly transformed by the matrix V :

$$\begin{aligned} [V^T (\partial\phi) V]_{mn} &= (V^T)_m{}^\mu (\partial_\mu \phi_\nu) V^\nu_n = \bar{\partial}_m \bar{\phi}_n, \\ \bar{\partial}_m &\equiv V^\mu{}_m \partial_\mu, \quad \bar{\phi}_m \equiv \phi_\mu V^\mu{}_m, \end{aligned} \quad (3.21)$$

so that

$$V^{-1} (\bar{g}^{-1} f) V = A^{(0)} (1 + ((\bar{\partial}\bar{\phi}) + (\bar{\partial}\bar{\phi})^T) + (\bar{\partial}\bar{\phi})(\bar{\partial}\bar{\phi})^T). \quad (3.22)$$

We should emphasize here that the derivatives ∂_μ are only acting on the Stückelberg field and *never* differentiate the “rotation matrix” elements $V^\mu{}_m$, even if the $V^\mu{}_m$ are written after ∂_μ .

Now the c-number part $A^{(0)}$ of this matrix is diagonal and its square root is simply given by

$$\sqrt{A^{(0)}}_{mn} = B^{(0)}_{mn} = \sqrt{\alpha(m)} \delta_{mn}. \quad (3.23)$$

It is more convenient to make the matrix (3.22) symmetric, so we further perform a similarity transformation by $B^{(0)}$, and call the resultant symmetric matrix A :

$$\begin{aligned} A &\equiv B^{(0)-1} V^{-1} (\bar{g}^{-1} f) V B^{(0)} \\ &= B^{(0)} (1 + ((\bar{\partial}\bar{\phi}) + (\bar{\partial}\bar{\phi})^T) + (\bar{\partial}\bar{\phi})(\bar{\partial}\bar{\phi})^T) B^{(0)} \\ &\equiv A^{(0)} + A^{(1)} + A^{(2)}. \end{aligned} \quad (3.24)$$

The matrices $A^{(1)}$ and $A^{(2)}$ are the linear and quadratic terms, respectively, in the Stückelberg field ϕ and their matrix elements are given more explicitly by

$$\begin{aligned} A^{(1)}_{mn} &= \sqrt{\alpha(m)} (\bar{\partial}_m \bar{\phi}_n + \bar{\partial}_n \bar{\phi}_m) \sqrt{\alpha(n)} = \bar{\bar{\partial}}_m \bar{\bar{\phi}}_n + \bar{\bar{\partial}}_n \bar{\bar{\phi}}_m, \\ A^{(2)}_{mn} &= \sqrt{\alpha(m)} \bar{\partial}_m \bar{\phi}_\ell \cdot \bar{\partial}_n \bar{\phi}_\ell \sqrt{\alpha(n)} = \bar{\bar{\partial}}_m \bar{\bar{\phi}}_\ell \cdot \bar{\bar{\partial}}_n \bar{\bar{\phi}}_\ell. \end{aligned} \quad (3.25)$$

Here the double-barred quantities $\bar{\bar{\phi}}$ and $\bar{\bar{\partial}}$ are defined as

$$\bar{\bar{\phi}}_m = \bar{\phi}_m \sqrt{\alpha(m)} = \phi_\mu V^\mu{}_m \sqrt{\alpha(m)} = \phi_\mu (V B^{(0)})^\mu{}_m, \quad (3.26)$$

and the same for $\bar{\bar{\partial}}_m$ with the understanding that the derivative acts only on ϕ but neither on $\sqrt{\alpha(m)}$ nor on $V^\mu{}_m$. Remember that the “vector” index of the barred quantities $\bar{\partial}$ and $\bar{\phi}$ defined in (3.21) now stands for the rotated one by V , and that of the double-barred quantities $\bar{\bar{\partial}}$ and $\bar{\bar{\phi}}$ for the “rotated” one by $V B^{(0)}$.

Before beginning a detailed computation, let us look at the “decoupling limit” at this stage. Our inspection of the expressions (3.25) finds it natural to define a decoupling limit by the following replacement, similar to the decoupling limit in the flat background case:

$$\bar{\phi}_m \rightarrow \bar{\bar{\phi}}_m \pi \quad \text{or, equivalently,} \quad \phi_\mu \rightarrow (\partial_\nu \pi) (V B^{(0)} V^{-1})^\nu{}_\mu. \quad (3.27)$$

It should be noted that the coefficients $(V B^{(0)} V^{-1})^\nu{}_\mu$ here must be *real* in order for this replacement to make sense. This is because ϕ_μ and $\partial_\nu \pi$ are real fields. Fortunately, from (3.17) and (3.23), we have $V A^{(0)} V^{-1} = \bar{g}^{-1} \eta$ and hence

$$V B^{(0)} V^{-1} = V \sqrt{A^{(0)}} V^{-1} = \sqrt{V A^{(0)} V^{-1}} = \sqrt{\bar{g}^{-1} \eta}, \quad (3.28)$$

so that $V B^{(0)} V^{-1}$ is a real matrix as long as $\sqrt{\bar{g}^{-1} \eta}$ is real. But the latter is the very condition that the present dRGT theory has a Hermitian mass term, so that it holds as long as the present theory makes sense.

We also note that this decoupling limit is quite nontrivial because it mixes time and spatial derivatives by the coefficients $(V B^{(0)} V^{-1})^\nu{}_\mu$ in general. This happens when the background metric \bar{g} has

a time–space component (shift). We will see this in more detail in an explicit example later. On the flat background $\bar{g} = \eta$, this of course reduces to the usual one $\phi_\mu \rightarrow \partial_\mu \pi$, and they are not mixed.

In this decoupling limit on the general background, we have

$$\begin{aligned}\bar{\partial}_m \bar{\phi}_n &\rightarrow \bar{\partial}_m \bar{\partial}_n \pi + (\partial_\nu \pi) [\bar{\partial}_m (V B^{(0)} V^{-1}) \cdot V]^\nu_n, \\ \bar{\partial}_m \bar{\phi}_n &\rightarrow (\bar{\partial}_m \bar{\partial}_n \pi) \sqrt{\alpha(n)} + (\partial_\nu \pi) [\bar{\partial}_m (V B^{(0)} V^{-1}) \cdot V B^{(0)}]^\nu_n,\end{aligned}\quad (3.29)$$

with $\bar{\partial}_m \bar{\partial}_n \pi$ denoting

$$\bar{\partial}_m \bar{\partial}_n \pi \equiv (\partial_\mu \partial_\nu \pi) (V B^{(0)})^\mu_m (V B^{(0)})^\nu_n. \quad (3.30)$$

That is, the derivative operator $\bar{\partial}_m$ here is understood to act only on the field π but not on the coefficients $(V B^{(0)})^\nu_n$, and then $\bar{\partial}_m$ and $\bar{\partial}_n$ are commutative on π . If we define a symmetric matrix Π by

$$\Pi_{mn} = \bar{\partial}_m \bar{\partial}_n \pi, \quad (3.31)$$

then, from Eq. (3.25), we have in this limit

$$\begin{aligned}A^{(1)}_{mn} &\rightarrow (B^{(0)} \Pi + \Pi B^{(0)})_{mn} + (\partial\pi\text{-term}), \\ A^{(2)}_{mn} &\rightarrow (\Pi^2)_{mn} + (\partial\pi\text{-term}),\end{aligned}\quad (3.32)$$

where $(\partial\pi\text{-term})$ denotes the first-order derivative terms of the π field. Namely, if we keep only the second-order derivative terms of π , neglecting the first-order derivative terms, then the matrix A takes a very simple form:

$$\begin{aligned}A &= A^{(0)} + A^{(1)} + A^{(2)} \\ &= (B^{(0)})^2 + (B^{(0)} \Pi + \Pi B^{(0)}) + \Pi^2 = (B^{(0)} + \Pi)^2.\end{aligned}\quad (3.33)$$

That is, as far as the second-order derivative terms $\partial\partial\pi$ are concerned,

$$B = \sqrt{A} = B^{(0)} + \Pi \quad (3.34)$$

in this decoupling limit and so there appear no quadratic terms of the Stückelberg field Π . This is a very similar situation to the flat background case, where actually it gave dRGT the motivation for taking the square root form for the mass term. This form (3.34) of \sqrt{A} guarantees that the dRGT mass terms generated by $\det[1 + \lambda\sqrt{A}]$ clearly have *total derivative* forms in the decoupling limit as far as the higher derivative terms $\partial\partial\pi$ are concerned.

Therefore, similarly to the flat case, we expect that the original Stückelberg “vector” field ϕ_μ appears only in the following “gauge-invariant” tensor combination in the quadratic terms in the mass term:

$$F_{mn} = \bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m. \quad (3.35)$$

This combination of $\bar{\partial}$ and $\bar{\phi}$ is suitable because of the form (3.27) of the decoupling limit $\bar{\phi}_\mu \rightarrow \bar{\partial}_\mu \pi$. We shall now show that this is indeed the case if we neglect some lower-order derivative terms.

3.2. Computing the general mass terms

Let us compute the generating function of the general mass terms:

$$\det[1 + \lambda\sqrt{\bar{g}^{-1}}f]. \quad (3.36)$$

Since this is invariant under the similarity transformation, we can use the expression A in (3.24) for the matrix $\sqrt{\bar{g}^{-1}}f$:

$$\det[1 + \lambda\sqrt{\bar{g}^{-1}}f] = \det[B^{(0)-1}V^{-1}(1 + \lambda\sqrt{\bar{g}^{-1}}f)VB^{(0)}] = \det[1 + \lambda\sqrt{A}]. \quad (3.37)$$

The square root of the matrix A can be calculated order by order in the Stückelberg field ϕ thanks to the fact that the matrix $B^{(0)}$ is diagonal. The matrix equation

$$B^{(0)} * X \equiv B^{(0)}X + XB^{(0)} = C \quad (3.38)$$

for X can be solved explicitly [23]. The solution X to this equation, denoted formally as $(B^{(0)*})^{-1}C$, is given explicitly by

$$X_{mn} = ((B^{(0)*})^{-1}C)_{mn} = \frac{1}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} C_{mn}. \quad (3.39)$$

This formula enables us to find the square root of A :

$$\sqrt{A^{(0)} + A^{(1)} + A^{(2)}}_{mn} = B^{(0)}_{mn} + B^{(1)}_{mn} + B^{(2)}_{mn} + \dots, \quad (3.40)$$

with

$$\begin{aligned} B^{(1)}_{mn} &= \frac{1}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} A^{(1)}_{mn}, \\ B^{(2)}_{mn} &= \frac{1}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} (A^{(2)}_{mn} - (B^{(1)}B^{(1)})_{mn}). \end{aligned} \quad (3.41)$$

Substituting the expression (3.25), we find

$$\begin{aligned} B^{(1)}_{mn} &= \frac{1}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} (\bar{\partial}\bar{\phi})_{(mn)}, \\ B^{(2)}_{mn} &= \frac{1}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} \sum_{\ell} \left\{ \bar{\partial}_m \bar{\phi}_{\ell} \cdot \bar{\partial}_n \bar{\phi}_{\ell} \right. \\ &\quad \left. - \frac{1}{(\sqrt{\alpha(m)} + \sqrt{\alpha(\ell)})(\sqrt{\alpha(n)} + \sqrt{\alpha(\ell)})} (\bar{\partial}\bar{\phi})_{(m\ell)} (\bar{\partial}\bar{\phi})_{(n\ell)} \right\}, \end{aligned} \quad (3.42)$$

with notation $(\bar{\partial}\bar{\phi})_{(mn)} \equiv \bar{\partial}_m \bar{\phi}_n + \bar{\partial}_n \bar{\phi}_m$.

Now we expand the determinant $\det[1 + \lambda\sqrt{A}] = \det[1 + \lambda B]$ in powers of the Stückelberg field ϕ :

$$\begin{aligned} \det[1 + \lambda B] &= \det[1 + \lambda(B^{(0)} + B^{(1)} + B^{(2)})] \\ &= \det[1 + \lambda B^{(0)}] \cdot \det[1 + \beta^{(1)} + \beta^{(2)}], \\ \beta^{(n)} &\equiv \frac{\lambda}{1 + \lambda B^{(0)}} B^{(n)}, \quad (n = 1, 2). \end{aligned} \quad (3.43)$$

The quadratic terms in ϕ is thus given by

$$\det[1 + \lambda B]|_{\text{quad}} = \det[1 + \lambda B^{(0)}] \cdot \left\{ \text{tr}[\beta^{(2)}] + \frac{1}{2} \left((\text{tr}[\beta^{(1)}])^2 - \text{tr}[(\beta^{(1)})^2] \right) \right\}. \quad (3.44)$$

We now simplify each term. First consider

$$\begin{aligned} \text{tr}[\beta^{(2)}] &= \sum_m \frac{\lambda}{1 + \lambda\sqrt{\alpha(m)}} \frac{1}{2\sqrt{\alpha(m)}} \sum_n \left\{ (\bar{\partial}_m \bar{\phi}_n)^2 - \frac{1}{(\sqrt{\alpha(m)} + \sqrt{\alpha(n)})^2} (\bar{\partial} \bar{\phi})_{(mn)}^2 \right\} \\ &= \frac{1}{2} \sum_{m,n} \frac{\lambda}{1 + \lambda\sqrt{\alpha(m)}} \frac{(\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m)}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} \left(2\bar{\partial}_m \bar{\phi}_n - \frac{\sqrt{\alpha(m)}(\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m)}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} \right). \end{aligned} \quad (3.45)$$

Averaging with the term obtained by exchanging the dummy indices $m \leftrightarrow n$, we get

$$\begin{aligned} \text{tr}[\beta^{(2)}] &= \frac{1}{4} \sum_{m,n} \frac{\lambda}{(1 + \lambda\sqrt{\alpha(m)})(1 + \lambda\sqrt{\alpha(n)})} \frac{(\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m)}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} \\ &\quad \times \left\{ \left(1 - \frac{2\lambda\sqrt{\alpha(m)\alpha(n)}}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} \right) (\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m) + 2\lambda(\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m) \right\}. \end{aligned} \quad (3.46)$$

The contribution of the second term in the bracket here is combined with the $\text{tr}[\beta^{(1)}\beta^{(1)}]$ term to yield

$$\begin{aligned} & -\frac{1}{2} \text{tr}[(\beta^{(1)})^2] + (\text{second term of Eq. (3.46)}) \\ &= -\frac{1}{2} \sum_{m,n} \frac{\lambda^2}{(1 + \lambda\sqrt{\alpha(m)})(1 + \lambda\sqrt{\alpha(n)})} \frac{1}{(\sqrt{\alpha(m)} + \sqrt{\alpha(n)})^2} \\ &\quad \times \left\{ (\bar{\partial}_m \bar{\phi}_n + \bar{\partial}_n \bar{\phi}_m)^2 - (\sqrt{\alpha(m)} + \sqrt{\alpha(n)})(\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m)(\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m) \right\} \\ &= \frac{1}{2} \sum_{m,n} \frac{\lambda^2}{(1 + \lambda\sqrt{\alpha(m)})(1 + \lambda\sqrt{\alpha(n)})} \frac{1}{(\sqrt{\alpha(m)} + \sqrt{\alpha(n)})^2} \\ &\quad \times \left\{ \sqrt{\alpha(m)\alpha(n)}(\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m)^2 - (\sqrt{\alpha(m)} + \sqrt{\alpha(n)})^2 \bar{\partial}_m \bar{\phi}_n \cdot \bar{\partial}_n \bar{\phi}_m \right\}, \end{aligned} \quad (3.47)$$

which partially cancels the first term in (3.46). We are thus left with

$$\begin{aligned} \text{tr}[\beta^{(2)}] - \frac{1}{2} \text{tr}[(\beta^{(1)})^2] &= \sum_{m,n} \frac{1}{(1 + \lambda\sqrt{\alpha(m)})(1 + \lambda\sqrt{\alpha(n)})} \\ &\quad \times \left\{ \frac{\lambda}{4} \frac{1}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} (\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m)^2 - \frac{\lambda^2}{2} \bar{\partial}_m \bar{\phi}_n \cdot \bar{\partial}_n \bar{\phi}_m \right\}. \end{aligned} \quad (3.48)$$

The first term takes a “gauge-invariant” form while the second term does not. The latter term is, however, almost “canceled” by the remaining term in (3.44):

$$+ \frac{1}{2} (\text{tr}[\beta^{(1)}])^2 = \frac{\lambda^2}{2} \sum_{m,n} \frac{1}{(1 + \lambda\sqrt{\alpha(m)})(1 + \lambda\sqrt{\alpha(n)})} (\bar{\partial}_m \bar{\phi}_m \cdot \bar{\partial}_n \bar{\phi}_n). \quad (3.49)$$

If we could do a partial integration with respect to the differential operators $\bar{\partial}_m$ and $\bar{\partial}_n$ here, this term would really cancel the second term in (3.48). But there are various x -dependent factors $\sqrt{\alpha(m)}$ and V^μ_m in front of the differential operators, the cancellation is not complete, and the terms with lower derivative terms of the form $\phi \partial \phi$ or $\phi \phi$ remain.

The final quadratic terms are thus given by

$$\begin{aligned} \det[1 + \lambda B]|_{\text{quad}} &= \prod_{\ell} (1 + \lambda \sqrt{\alpha(\ell)}) \cdot \sum_{m,n} \frac{1}{(1 + \lambda \sqrt{\alpha(m)})(1 + \lambda \sqrt{\alpha(n)})} \\ &\times \left\{ \frac{\lambda}{\sqrt{\alpha(m)} + \sqrt{\alpha(n)}} \frac{1}{4} (\bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m)^2 + \frac{\lambda^2}{2} (\bar{\partial}_m \bar{\phi}_m \cdot \bar{\partial}_n \bar{\phi}_n - \bar{\partial}_m \bar{\phi}_n \cdot \bar{\partial}_n \bar{\phi}_m) \right\}. \end{aligned} \quad (3.50)$$

3.3. Gauge invariance and the no-ghost theorem

As anticipated from the consideration of the decoupling limit, the resultant generic mass term is almost “gauge invariant” under

$$\delta \bar{\phi}_m = \bar{\partial}_m \Lambda, \quad \text{or, more precisely,} \quad \delta \phi_\mu = (\partial_\nu \Lambda) (V B^{(0)} V^{-1})^\nu{}_\mu. \quad (3.51)$$

Actually, it is not exactly invariant since the coefficients V^μ_m and $\sqrt{\alpha(m)}$ are x -dependent and the derivatives do not commute with them. So we find that it is convenient to introduce the Stückelberg “scalar” field π by

$$\phi_\mu = A_\mu + (\partial_\nu \pi) (V B^{(0)} V^{-1})^\nu{}_\mu. \quad (3.52)$$

Then the $U(1)$ gauge invariance under

$$\delta A_\mu = (\partial_\nu \Lambda) (V B^{(0)} V^{-1})^\nu{}_\mu \quad \text{and} \quad \delta \pi = -\Lambda \quad (3.53)$$

becomes exact since the change cancels between the A_μ and $\partial\pi$ terms, leaving ϕ_μ intact. It is important to make this $U(1)$ gauge invariance exact; this is because it is lifted to the BRST invariance to define the physical subspace in covariant gauges so that it must be an *exact gauge symmetry* of the total action.

The above-mentioned *approximate “gauge invariance”* under (3.51), on the other hand, guarantees that the higher derivative terms in the kinetic term of the π field cancel. This is essentially due to the fact that the Stückelberg field expression (3.52) for ϕ_μ is defined in accordance with the decoupling limit (3.27).

Let us now explicitly show that the higher derivative terms of the π field indeed cancel in the kinetic term (3.50).

First, consider the first term in (3.50) written in terms of $F_{mn} = \bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m$. Note that the Stückelberg expression (3.52) for ϕ gives

$$\bar{\partial}_m \bar{\phi}_n = \bar{\partial}_m \bar{A}_n + \bar{\partial}_m \bar{\partial}_n \pi + C_{mn}^\rho \partial_\rho \pi, \quad (3.54)$$

where $\bar{\partial}_m \bar{\partial}_n \pi$ is defined in (3.30) and the coefficient C_{mn}^ρ of the $\partial\pi$ term is given by

$$C_{mn}^\rho = [\bar{\partial}_m (V B^{(0)} V^{-1}) \cdot V]^\rho{}_n. \quad (3.55)$$

Recalling that $\bar{\partial}_m \bar{\partial}_n \pi$ defined in (3.30) is symmetric under $m \leftrightarrow n$, we see that the second-order derivative terms $\bar{\partial} \bar{\partial} \pi$ cancel in

$$F_{mn} \equiv \bar{\partial}_m \bar{\phi}_n - \bar{\partial}_n \bar{\phi}_m = (\bar{\partial}_m \bar{A}_n - \bar{\partial}_n \bar{A}_m) + (C_{mn}^\rho - C_{nm}^\rho) \partial_\rho \pi, \quad (3.56)$$

so that the first term in (3.50) contains only the first-order derivative $\partial\pi$ of the π field.

Next, consider the second term in (3.50). In order to do the partial integration carefully, let us make explicit the factors contained in the definitions of barred quantities:

$$\bar{X}_m = X_\mu V^\mu{}_m, \quad \bar{\bar{X}}_m = X_\mu (V B^{(0)})^\mu{}_m. \quad (3.57)$$

We define the coefficient $C^{\mu\nu}$, which will frequently appear below:

$$C^{\mu\nu} \equiv (V B^{(0)} V^T)^{\mu\nu} = C^{\nu\mu}. \quad (3.58)$$

Noting $V^{-1} = V^T \eta$, we can rewrite the Stückelberg field expression (3.52) in the form

$$\phi_\mu = A_\mu + (\partial_\nu \pi) (V B^{(0)} V^T)^{\nu\rho} \eta_{\rho\mu} = A_\mu + \eta_{\mu\rho} C^{\rho\nu} \partial_\nu \pi. \quad (3.59)$$

We find

$$\bar{\partial}_m \bar{\phi}_m = \partial_\rho \phi_\sigma (V B^{(0)})^\rho{}_m V^\sigma{}_m = \partial_\rho \phi_\sigma (V B^{(0)} V^T)^{\rho\sigma} = C^{\rho\sigma} \partial_\rho \phi_\sigma, \quad (3.60)$$

and, similarly,

$$\bar{\partial}_m \bar{\phi}_m \cdot \bar{\partial}_n \bar{\phi}_n - \bar{\partial}_m \bar{\phi}_n \cdot \bar{\partial}_n \bar{\phi}_m = C^{\mu\alpha} C^{\nu\beta} (\partial_\mu \phi_\alpha \cdot \partial_\nu \phi_\beta - \partial_\mu \phi_\beta \cdot \partial_\nu \phi_\alpha). \quad (3.61)$$

Consequently, the second term in (3.48) can be put, after performing partial integrations twice, into the form

$$c(\bar{\partial}_m \bar{\phi}_m \cdot \bar{\partial}_n \bar{\phi}_n - \bar{\partial}_m \bar{\phi}_n \cdot \bar{\partial}_n \bar{\phi}_m) = \phi_\alpha \partial_\mu \partial_\nu (c C^{\mu\alpha} C^{\nu\beta}) \cdot \phi_\beta + 2\phi_\alpha \partial_\nu (c C^{\mu\alpha} C^{\nu\beta}) \cdot \partial_\mu \phi_\beta, \quad (3.62)$$

where c stands for all the prefactors in front of this term in the action (including $\det \sqrt{g}$). Now the first term on the right-hand side of (3.62) contains only ϕ s with no derivatives, so that it contains at most first-order derivatives of the π fields. The second term seems to contain $\partial\phi$, which gives the second-order derivative of π since

$$\partial_\mu \phi_\beta = \partial_\mu A_\beta + \partial_\mu (\eta_{\beta\rho} C^{\rho\nu} \partial_\nu \pi). \quad (3.63)$$

Nevertheless, we now show that these second-order derivative terms of π vanish. Since the first-order derivative of the “vector” field A_μ is in any case contained in the action, we can forget about it here. Keeping only the π field in ϕ , we find that the second term of (3.62) becomes

$$\begin{aligned} & 2\phi_\alpha \partial_\nu (c C^{\mu\alpha} C^{\nu\beta}) \cdot \partial_\mu \phi_\beta |_{\pi^2 \text{ terms}} \\ &= 2\eta_{\alpha\delta} C^{\delta\rho} \partial_\rho \pi \cdot \partial_\nu (c C^{\mu\alpha} C^{\nu\beta}) \cdot (\eta_{\beta\gamma} \partial_\mu C^{\gamma\tau} \cdot \partial_\tau \pi + \eta_{\beta\gamma} C^{\gamma\tau} \cdot \partial_\mu \partial_\tau \pi). \end{aligned} \quad (3.64)$$

The first term is harmless with only the first derivatives on the π s, but the last term is the dangerous one containing the second derivative $\partial\partial\pi$, which we write in the form

$$\text{the last term of (3.64)} = 2d^{\mu\nu\rho} \partial_\nu \pi \cdot \partial_\mu \partial_\rho \pi \equiv L, \quad (3.65)$$

by introducing a coefficient

$$d^{\mu\nu\rho} \equiv (\eta C)_\alpha{}^\nu (\eta C)_\beta{}^\rho \partial_\gamma (c C^{\mu\alpha} C^{\gamma\beta}). \quad (3.66)$$

By performing a partial integration for ∂_μ , we can rewrite (3.65) as

$$L = -2d^{\mu\nu\rho} \partial_\mu \partial_\nu \pi \cdot \partial_\rho \pi - 2(\partial_\mu d^{\mu\nu\rho}) \partial_\nu \pi \cdot \partial_\rho \pi. \quad (3.67)$$

Averaging these two expressions (3.65) and (3.67), we have

$$L = (d^{\mu\rho\nu} - d^{\mu\nu\rho}) \partial_\mu \partial_\nu \pi \cdot \partial_\rho \pi - (\partial_\mu d^{\mu\nu\rho}) \partial_\nu \pi \cdot \partial_\rho \pi. \quad (3.68)$$

Noticing that the coefficient of the first term $d^{\mu\rho\nu} - d^{\mu\nu\rho} \equiv 2d^{\mu[\rho\nu]}$ is antisymmetric under $\rho \leftrightarrow \nu$, we can make a partial integration to put it into the first-order derivative terms:

$$\begin{aligned} L &= -2(\partial_\nu d^{\mu[\rho\nu]})\partial_\mu\pi \cdot \partial_\rho\pi - (\partial_\mu d^{\mu\nu\rho})\partial_\nu\pi \cdot \partial_\rho\pi \\ &= (\partial_\nu(d^{\mu\nu\rho} - d^{\mu\rho\nu} - d^{\nu\mu\rho}))\partial_\mu\pi \cdot \partial_\rho\pi. \end{aligned} \quad (3.69)$$

We have thus shown that all the π field terms can be put solely into first-order derivative terms. So the quadratic part in the fields of the mass term takes the usual form $L(\varphi, \partial\varphi)$, containing only up to first-order derivatives for all the fields $\varphi = \{h_{\mu\nu}, A_\mu, \pi\}$.¹

The $U(1)$ gauge invariance is exact and all the fields appear only up to the first-order derivative in the quadratic kinetic term. On any background metric, the particle modes are determined by the quadratic terms. Combined with our previous counting of physical degrees of freedom, this implies that there appears no BD ghost mode in this theory on the general background metric, and completes our proof of the no-ghost theorem.

4. Discussions

It is instructive to see explicitly the general result in the previous section for a concrete nontrivial background example. Let us consider the following background metric $\bar{g}_{\mu\nu}$, discussed by dRGT [10]:

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \delta_{ij}(dx^i + 2l^i dt)(dx^j + 2l^j dt). \quad (4.1)$$

This is the metric with the lapse $N = 1$ and the shift vector $N^i = 2l^i$. Since the space metric γ_{ij} is taken to be δ_{ij} , we can freely rotate the spatial axis such that the shift vector points in the x^1 direction:

$$\delta_{ij}l^i dx^j = l dx^1. \quad (4.2)$$

For this background metric $\bar{g}_{\mu\nu}$, we have

$$(\bar{g}^{-1}\eta)^\mu{}_\nu = \begin{pmatrix} 1 & 2l & & \\ -2l & 1 - 4l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad (4.3)$$

where the blank entries are all zeros. The characteristic equation for the first nontrivial 2×2 matrix in the (x^0, x^1) subspace is

$$x^2 - 2(1 - l^2)x + 1 = 0. \quad (4.4)$$

The metric is flat for $l = 0$. For the reason to become clear shortly, we consider only the case $|l| < 1$. The eigenvalues are then complex:

$$\begin{cases} \alpha(0) = \alpha \\ \alpha(1) = \alpha^* \end{cases} \quad \text{with} \quad \alpha = (\sqrt{1 - l^2} + il)^2. \quad (4.5)$$

The eigenvectors for these two eigenvalues in the (x^0, x^1) subspace are conveniently chosen as

$$V_{(2)} = (V_1, V_2) = \frac{1}{N} \begin{pmatrix} -ia^* & ia \\ a & a^* \end{pmatrix} \quad \text{with} \quad \begin{aligned} a &\equiv \sqrt{\sqrt{1 - l^2} + il} = \sqrt[4]{\alpha} \\ N &\equiv \sqrt{2\sqrt{1 - l^2}} \end{aligned} \quad (4.6)$$

¹ Although we have set $h_{\mu\nu} = 0$ in this calculation, it is clear that $h_{\mu\nu}$ appears only without derivatives in the mass term, so that it can appear in the quadratic term in the form $h\partial\phi$ at the highest derivative order. $h\partial\phi$ contains the second-order derivative of π , $h\partial\partial\pi$, but it can be rewritten into the first-order derivative term $\partial h \cdot \partial\pi$.

Note that α and a are unimodular: $\alpha\alpha^* = 1 = aa^*$, and satisfy

$$\sqrt{\alpha} + \sqrt{\alpha^*} = 2\sqrt{1-l^2}, \quad i(\sqrt{\alpha} - \sqrt{\alpha^*}) = -2l, \quad \alpha + \alpha^* = 2(1-2l^2). \quad (4.7)$$

The other two eigenvalues and eigenvectors in the (x^2, x^3) directions are trivial. Hence the matrix V that diagonalizes the matrix $\bar{g}^{-1}\eta$ in (4.3) and the diagonalized matrix are given by

$$V = \begin{pmatrix} V_{(2)} & \\ & 1_2 \end{pmatrix} \rightarrow V^{-1}\bar{g}^{-1}\eta V = \begin{pmatrix} \alpha & & & \\ & \alpha^* & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \quad (4.8)$$

Note that this matrix V is properly normalized so as to satisfy Eq. (3.18):

$$V^{-1} = V^T \eta. \quad (4.9)$$

Now the barred derivatives $\bar{\partial}_m = V^\mu{}_m \partial_\mu$ defined in Eq. (3.21) are explicitly read as

$$\begin{aligned} \bar{\partial}_0 &= \frac{1}{N} (-ia^* \partial_0 + a \partial_1), & \bar{\bar{\partial}}_0 &= \sqrt{\alpha} \bar{\partial}_0, \\ \bar{\partial}_1 &= \frac{1}{N} (ia \partial_0 + a^* \partial_1) = \bar{\partial}_0^*, & \bar{\bar{\partial}}_1 &= \sqrt{\alpha^*} \bar{\partial}_1 = \bar{\bar{\partial}}_0^*, \end{aligned} \quad (4.10)$$

and, of course, $\bar{\partial}_2 = \partial_2$, $\bar{\partial}_3 = \partial_3$. The barred fields $\bar{\phi}_m = V^\mu{}_m \phi_\mu$ are similar:

$$\begin{aligned} \bar{\phi}_0 &= \frac{1}{N} (-ia^* \phi_0 + a \phi_1), & \bar{\bar{\phi}}_0 &= \sqrt{\alpha} \bar{\phi}_0, \\ \bar{\phi}_1 &= \frac{1}{N} (ia \phi_0 + a^* \phi_1) = \bar{\phi}_0^*, & \bar{\bar{\phi}}_1 &= \sqrt{\alpha^*} \bar{\phi}_1 = \bar{\bar{\phi}}_0^*. \end{aligned} \quad (4.11)$$

Our result for the general mass term $\det[1 + \lambda B]|_{\text{quad}}$ was given in Eq. (3.50). If we keep only the nontrivial terms $\bar{\partial}_m \bar{\phi}_n$ with $(m, n) = (1, 0)$ and $(0, 1)$, it gives

$$\begin{aligned} \det[1 + \lambda B]|_{\text{quad}} &= (1 + \lambda)^2 \left\{ \frac{\lambda}{\sqrt{\alpha} + \sqrt{\alpha^*}} \frac{1}{2} (\sqrt{\alpha} \bar{\partial}_0 \bar{\phi}_1 - \sqrt{\alpha^*} \bar{\partial}_1 \bar{\phi}_0)^2 \right. \\ &\quad \left. + \lambda^2 \alpha \alpha^* (\bar{\partial}_0 \bar{\phi}_0 \cdot \bar{\partial}_1 \bar{\phi}_1 - \bar{\partial}_0 \bar{\phi}_1 \cdot \bar{\partial}_1 \bar{\phi}_0) \right\}. \end{aligned} \quad (4.12)$$

Substituting Eqs. (4.10) and (4.11), and using Eq. (4.7), we find that this reduces to

$$(1 + \lambda)^2 \left\{ \frac{\lambda}{4(1-l^2)^{3/2}} (\dot{\phi}_1 - l\dot{\phi}_0 + (2l^2 - 1)\phi'_0 - l\phi'_1)^2 + \lambda^2 (\dot{\phi}_1 \phi'_0 - \dot{\phi}_0 \phi'_1) \right\}, \quad (4.13)$$

where $\dot{\phi} \equiv \partial_0 \phi$, $\phi' \equiv \partial_1 \phi$. Note that the second term has lost the x^μ -dependent coefficients and the overall factor $\sqrt{-\bar{g}} = 1$ in front is also x^μ -independent, so the second term can be partial-integrated away. Note also that the first term contains the square of $\dot{\phi}_0$, which would yield the square of the second-order time derivative $\ddot{\pi}$ if we had introduced the Stückelberg scalar field π in the same manner as the flat background case:

$$\phi_\mu \rightarrow \partial_\mu \pi. \quad (4.14)$$

As was argued in Ref. [10], this term is actually harmless because $\dot{\phi}_0$ comes into the action only with the particular combination $(\dot{\phi}_1 - l\dot{\phi}_0)$ with $\dot{\phi}_1$ and does not give rise to another degree of freedom

than ϕ_1 . In our discussions, we can see the absence of the ghost in a better way. It is important to remember that the proper way of introducing π in the general background is not (4.14) but

$$\phi_\mu \rightarrow \partial_\nu \pi (V B^{(0)} V^{-1})^\nu{}_\mu, \quad (4.15)$$

as given in Eq. (3.52). The coefficient $(V B^{(0)} V^{-1})^\nu{}_\mu$ reads

$$V B^{(0)} V^{-1} = \sqrt{\bar{g}^{-1} \eta} = \begin{pmatrix} \frac{1}{1-l^2} \begin{pmatrix} 1 & l \\ -l & 1-2l^2 \end{pmatrix} & \\ & 1 \\ & & 1 \end{pmatrix}, \quad (4.16)$$

which is indeed real, as it should be. Therefore our definition of the π field yields

$$\phi_0 \rightarrow \frac{\dot{\pi} - l\pi'}{1-l^2}, \quad \phi_1 \rightarrow \frac{l\dot{\pi} + (1-2l^2)\pi'}{1-l^2}. \quad (4.17)$$

If we substitute this into the first term in (4.13) and concentrate on the second-order derivative terms of π (forgetting about the terms with the coefficients differentiated), we have

$$\begin{aligned} & \dot{\phi}_1 - l\dot{\phi}_0 + (2l^2 - 1)\phi'_0 - l\phi'_1 \\ &= \frac{l\ddot{\pi} + (1-2l^2)\dot{\pi}'}{1-l^2} - l\frac{\ddot{\pi} - l\pi''}{1-l^2} + (2l^2 - 1)\frac{\dot{\pi}' - l\pi''}{1-l^2} - l\frac{l\dot{\pi}' + (1-2l^2)\pi''}{1-l^2} = 0! \end{aligned} \quad (4.18)$$

Thus we explicitly see that all the second-order derivative terms of π disappear, as was shown generally in the previous section. This is due to the “gauge invariance” of the F_{mn} term under $\delta\bar{\phi}_m = \bar{\partial}_m\pi$. This also clearly shows the importance and nontriviality of our definition of the Stückelberg π field or decoupling limit in the general curved spacetime.

When l becomes 1, our expression for the quadratic term of the mass term diverges (see Eq. (4.13)). What happens there?

As long as the condition $l^2 < 1$ is satisfied, the characteristic equation (4.4) has two roots α and α^* , and the matrix $\bar{g}^{-1}\eta$ is diagonalizable. But when l becomes as large as 1, the complex eigenvalues α and α^* become degenerate and take the value -1 , and the corresponding eigenvectors V_1 and V_2 also degenerate, i.e., $NV_1 \propto NV_2$. This implies that the eigenvectors do not span a complete set, so that the matrix $\bar{g}^{-1}\eta$ is non-diagonalizable. At $l = 1$, $\bar{g}^{-1}\eta$ can be brought at most into a Jordan standard form:

$$V^{-1}\bar{g}^{-1}\eta V = \begin{pmatrix} -1 & 1 & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \quad (4.19)$$

The form of the quadratic kinetic term for the Stückelberg fields, which was derived in the previous section assuming diagonalizability, diverges in the limit $l \rightarrow 1$.

Fortunately, the mass term $\frac{1}{2}[(K_\mu^\mu)^2 - K_\mu^\nu K_\nu^\mu]$ can be calculated exactly in this example if we retain only the $\partial_\mu\phi_\nu$ terms with $\mu, \nu = 0, 1$.² This is fine since we are mainly interested in the time

² In this case, the matrix $A = \bar{g}^{-1}f$ becomes essentially 2D. Any 2×2 real matrix A can always be written in the form $A = a_0 1_2 + \vec{a} \cdot \vec{\sigma}$ in terms of four parameters a_μ , three real a_0, a_1, a_3 and one purely imaginary a_2 , together with the unit matrix 1_2 and the Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. Using this parametrization and the

derivatives of the fields. If we keep only the time derivative terms $\dot{\phi}_0$ and $\dot{\phi}_1$, we find

$$2 - \dot{\phi}_0 - \sqrt{4(1 - l^2) - 4(\dot{\phi}_0 - l\dot{\phi}_1) + \dot{\phi}_0^2 - \dot{\phi}_1^2}. \quad (4.21)$$

If we look at Eq. (4.21) for $l = 1$, we see that the $\dot{\phi}_\mu = 0$ point becomes the branch point of the square root, so that the expansion itself of the mass term in powers of the Stückelberg fields ϕ does not make sense.

The origin of the square root here is, of course, the square root factor $\sqrt{g^{-1}f}$ of the dRGT mass term. So even if we do not introduce the Stückelberg fields ϕ (i.e., setting $f_{\mu\nu} = \eta_{\mu\nu}$), this singularity at $g = \bar{g}$ with $l = 1$ is the singularity of the Lagrangian itself and the metric fluctuation $h_{\mu\nu}$ around the background $g = \bar{g}$ does not make sense. This does not allow for any particle interpretation.

Also, beyond $|l| = 1$, the background value inside the square root in Eq. (4.21) is negative, and again this implies that the square root factor $\sqrt{g^{-1}f}$ in the dRGT mass term comes to have a *complex* value at the background $g = \bar{g}$ so that the dRGT Lagrangian itself becomes *non-Hermitian* and no longer gives a well defined theory.

This is the reason why we have to restrict the shift vector to $|l| < 1$; in this region our discussions work perfectly well and there is no ghost in this massive gravity. This must be the general situation; as long as the dRGT mass term defines a Hermitian Lagrangian, then the matrix $\bar{g}^{-1}f$ is diagonalizable and the general no-ghost proof in the previous section will apply.

In summary, we have discussed the no-ghost theorem in massive gravity. We start with a discussion of the simple gravity theory with the Fierz–Pauli mass term and analyze the spectrum in a covariant manner. Naively we have six degrees of freedom since the general coordinate invariance is broken in the presence of the mass term. However, we have shown that one of the modes, the BD ghost, decouples for the special choice of the mass term. By introducing the Stückelberg fields, which recover the general coordinate invariance, and using the BRST formalism, we have then clarified how the various modes in the theory cancel each other, leaving the correct five degrees of freedom. The crucial point in this formulation is that there remains no higher (time) derivative on the Stückelberg fields.

We then proceed to a discussion of the nonlinear dRGT massive gravity on arbitrary backgrounds. Because of the complicated nature of the square root form of the mass term, it is rather cumbersome to identify fluctuations around arbitrary backgrounds, but we were able to do it by diagonalizing the background. We have then shown that there remains no higher (time) derivatives on the Stückelberg fields, and hence the theory is ghost free. In this process, we have identified the correct way to introduce the Stückelberg fields on general backgrounds, and found that the associated decoupling limit is also quite nontrivial, naively mixing time and space derivatives. Nonetheless, we have shown that this does not cause trouble with the ghost. Rather, this is necessary in order for the ghost to decouple. This is further confirmed by an explicit example.

Recently, it has been shown that this class of massive gravity can be derived from the five-dimensional Einstein gravity by deconstruction [24]. It would be interesting to extend that approach to supergravity and study the structure of the theory.

properties of the Pauli matrices, one can easily find the square root as

$$\sqrt{A} = b_0 1_2 + \frac{1}{2b_0} \vec{a} \cdot \vec{\sigma} \quad \text{with} \quad 2b_0^2 = \frac{1}{2} \text{tr} A + \sqrt{\det A}. \quad (4.20)$$

Acknowledgements

Part of this work was carried out while the authors were attending the molecule-type workshop “Nonlinear massive gravity theory and its observational test” (ID: YITP-T-12-04). We thank the organizer (Tetsuya Shiromizu) and participants, in particular Claudia de Rham and Andrew Tolley, for stimulating discussions. Thanks are also due to Cedric Deffayet, Fawad Hassan, Kei-ichi Maeda, Yuho Sakatani, and Tomohiko Takahashi for valuable discussions. We are especially indebted to Shinji Mukohyama who taught us the expansion formula for the matrix square root around the diagonal matrix, which was crucial for the present work. One of the authors (T.K.) is partially supported by a Grant-in-Aid for Scientific Research (B) (No. 24340049) from the Japan Society for the Promotion of Science (JSPS). The other author (N.O.) is supported in part by the Grant-in-Aid for the Scientific Research Fund of the JSPS (C) No. 24540290, and (A) No. 22244030.

Funding

Open Access funding: SCOAP³.

References

- [1] M. Fierz and W. Pauli, Proc. R. Soc. Lond. A **173**, 211 (1939).
- [2] H. van Dam and M. J. G. Veltman, Nucl. Phys. B **22**, 397 (1970).
- [3] V. I. Zakharov, JETP Lett. **12**, 312 (1970) [Pisma Zh. Eksp. Teor. Fiz. **12**, 447 (1970)].
- [4] A. I. Vainshtein, Phys. Lett. B **39**, 393 (1972).
- [5] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).
- [6] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]].
- [7] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011) [arXiv:1011.1232 [hep-th]].
- [8] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Ann. Phys. **305**, 96 (2003) [arXiv:0210184 [hep-th]].
- [9] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. **108**, 041101 (2012) [arXiv:1106.3344 [hep-th]].
- [10] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Lett. B **711**, 190 (2012) [arXiv:1107.3820 [hep-th]].
- [11] C. de Rham, G. Gabadadze, and A. J. Tolley, J. High Energy Phys. **1111**, 093 (2011) [arXiv:1108.4521 [hep-th]].
- [12] S. F. Hassan, R. A. Rosen, and A. Schmidt-May, J. High Energy Phys. **1202**, 026 (2012) [arXiv:1109.3230 [hep-th]].
- [13] S. F. Hassan and R. A. Rosen, J. High Energy Phys. **1204**, 123 (2012) [arXiv:1111.2070 [hep-th]].
- [14] M. Mirbabayi, Phys. Rev. D **86**, 084006 (2012) [arXiv:1112.1435 [hep-th]].
- [15] S. F. Hassan, A. Schmidt-May, and M. von Strauss, Phys. Lett. B **715**, 335 (2012) [arXiv:1203.5283 [hep-th]].
- [16] K. Hinterbichler and R. A. Rosen, J. High Energy Phys. **1207**, 047 (2012) [arXiv:1203.5783 [hep-th]].
- [17] C. Deffayet, J. Mourad, and G. Zahariade, J. Cosmol. Astropart. Phys. **1301**, 032 (2013) [arXiv:1207.6338 [hep-th]].
- [18] K. Hinterbichler, J. High Energy Phys. **1310**, 102 (2013) [arXiv:1305.7227 [hep-th]].
- [19] R. Kimura and D. Yamauchi, Phys. Rev. D **88**, 084025 (2013) [arXiv:1308.0523 [gr-qc]].
- [20] C. de Rham, A. Matas, and A. J. Tolley, arXiv:1311.6485 [hep-th].
- [21] N. Ohta, Classical Quantum Gravity **29**, 015002 (2012) [arXiv:1109.4458 [hep-th]].
- [22] S. Hamamoto, Prog. Theor. Phys. **114**, 1261 (2006) [arXiv:0505194 [hep-th]].
- [23] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, J. Cosmol. Astropart. Phys. **1203**, 006 (2012) [arXiv:1111.4107 [hep-th]].
- [24] C. de Rham, A. Matas, and A. J. Tolley, Classical Quantum Gravity **31**, 025004 (2014) [arXiv:1308.4136 [hep-th]].